A Mathematical Model of Deforming Manifolds and Their Visualizations by CG Animation

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It is becoming an important field in computer art to visualize High Dimensional Manifolds. [Banchoff 1990] [Kusabuka and AlgorithmicArt 2006] In this research, we will find the dynamics equation which express deforming motion of the α dimensional manifold r , and visualize many interesting motions of the deformed manifold $r(t)$ by CG animation.

Let r be the deformed manifold and s be the natural configuration of the manifold, which is embedded in the general dimensional Euclidean space $\mathbf{R}^{\alpha+1}$. The manifolds r and s are covered by plural charts, i.e. they equal the union sets of some plural charts. They are parameterized by $r(u) = r(u_1, u_2, \cdots, u_\alpha)$ and $s(u) = s(u_1, u_2, \cdots, u_\alpha)$ on each chart, where the local coordinate $u = (u_1, u_2, \cdots, u_\alpha)$ is a variable in α dimensional Euclidean space $\mathbf{R}^\alpha$. They also be parameterized by $r(v) = r(v_1, v_2, \cdots, v_\alpha)$ and $s(v) = s(v_1, v_2, \cdots, v_\alpha)$ on another chart.

Let $M(u)$ be the normal vector of the deformed manifold $r(u)$, and $N(u)$ be the normal vector of the natural manifold $s(u)$. Let $\langle \cdot , \cdot \rangle$ denote the inner product (the dot product) in $(\alpha + 1)$ dimensional Euclidean space $\mathbf{R}^{\alpha+1}$. Let f_{ij} be defined as

$$f_{ij} = \left\langle \frac{\partial^2 r}{\partial u_i \partial u_j}, M \right\rangle - \left\langle \frac{\partial^2 s}{\partial u_i \partial u_j}, N \right\rangle \quad (1)$$

Since the difference between the 2nd fundamental forms of the deformed manifold r and the natural manifold s

$$\text{Difference}_{(\text{2nd})}(r, s) = -\sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha} f_{ij} du_i du_j \quad (2)$$

is conserved by the transformation from a local coordinate $u = (u_1, u_2, \cdots, u_\alpha)$ to another local coordinate $v = (v_1, v_2, \cdots, v_\alpha)$, the potential function

$$J_{(\text{2nd})}(r, s) = \int du_i du_j \sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha} \left\{ \left\langle \frac{\partial^2 r}{\partial u_i \partial u_j}, M \right\rangle - \left\langle \frac{\partial^2 s}{\partial u_i \partial u_j}, N \right\rangle \right\}^2 \quad (3)$$

can be defined as an integral over the manifold. This potential function $J_{(\text{2nd})}(r, s)$ expresses the amount of differences between the configuration of the deformed manifold r and the natural manifold s with respect to their bend and torsion.

The Frechet derivative $\partial J_{(\text{2nd})}(r, s)/\partial r$ of the potential function $J_{(\text{2nd})}(r, s)$ with respect to r becomes

$$\frac{\partial J_{(\text{2nd})}(r, s)}{\partial r} = 2\sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha}\sum_{k=1}^{\alpha}\sum_{m=1}^{\alpha}\sum_{l=1}^{\alpha+1} \vec{e}_l (-1)^l \det\left(l \text{ th raw is omitted from } c_{km}\right) + 2\sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha} \frac{\partial^2}{\partial u_i \partial u_j}(f_{ij} M) \quad (4)$$

where $\vec{e}_l$ is an $(\alpha + 1)$ dimensional raw vector whose l th element is 1 and all other elements are 0 , and the matrix c_{km} is defined as follows. For $m = k$,

$$c_{km} = \left(\frac{\partial r}{\partial u_1}, \cdots, \frac{\partial r}{\partial u_{k-1}}, \frac{\partial}{\partial u_k}\left(f_{ij} \frac{\partial^2 r}{\partial u_i \partial u_j} \right), \frac{\partial r}{\partial u_{k+1}}, \cdots, \frac{\partial r}{\partial u_\alpha} \right) \quad (5)$$

For $m \neq k$, n th column c_{kmn} of the matrix c_{km} is obtained by

$$c_{kmn} = \frac{\partial}{\partial u_k}\left(\frac{\partial r}{\partial u_m}\right) \quad \text{for} \quad n = m \quad (6)$$

$$c_{kmn} = f_{ij} \frac{\partial^2 r}{\partial u_i \partial u_j} \quad \text{for} \quad n = k \quad (7)$$

$$c_{kmn} = \frac{\partial r}{\partial u_n} \quad \text{for} \quad n \neq m \quad \text{and} \quad n \neq k \quad (8)$$

The above equation (4) generates the inner forces of the manifold r with respect to bend and torsion.

We can also define the potential function $J_{(\text{1st})}(r, s)$ from the difference $\text{Difference}_{(\text{1st})}(r, s)$ between the 1st fundamental forms of the deformed manifold r and the natural manifold s , and the Frechet derivative $\partial J_{(\text{1st})}(r, s)/\partial r$ of this potential function $J_{(\text{1st})}(r, s)$ with respect to r becomes

$$\frac{\partial J_{(\text{1st})}(r, s)}{\partial r} = -4\sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha} \frac{\partial}{\partial u_j}\left(g_{ij} \frac{\partial r}{\partial u_i} \right) \quad (9)$$

This Equation (9) generates the inner forces of the manifold r with respect to stretch and shear.

Let $\rho(u)$ be the mass density ot the deformed manifold r where $u = (u_1, u_2, \cdots, u_\alpha)$ is a local coordinate of some chart. Let $r(t)$ be the configure of the deformed manifold r at time t. The motion equation of the deforming manifold $r(t)$ becomes

$$\rho(u) \frac{\partial^2 r(t, u)}{\partial t^2} = -\frac{\partial J_{(\text{2nd})}}{\partial r}(r(t, u), s(u)) - \frac{\partial J_{(\text{1st})}}{\partial r}(r(t, u), s(u)) \quad (10)$$

In the real world, the natural configuration s of the manifold does not change as the time t goes on. But, in the virtual world, we can change the natural configuration s as the time t goes on, i.e. we can consider time time varying natural configuration $s(t, u)$. Many interesting motions of the deformed manifold $r(t, u)$ are obtained from the appropriate time varying natural configuration $s(t, u)$.

References

BANCHOFF, T. F. 1990. *Beyond The Third Dimension – Geometry, Computer Graphics and Higher Dimensions –*. Scientific American Library.

KUSABUKA, K., AND ALGORITHMICART, S. 2006. *20 th Century Computer Art : Beginning and Development – The work and thought of pioneers and contemporary practitioners of algorithmic art –*. Tama Art University Museum.

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SIGGRAPH 2010, Los Angeles, California, July 25 – 29, 2010.
ISBN 978-1-4503-0210-4/10/0007